

Maximum Sustainable Government Debt in the Perpetual Youth Model*

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Abstract

The overlapping-generations model of Blanchard, based on a constant probability of death, is used to study the maximum level of government debt consistent with the existence of a steady state equilibrium. In both a small open and a closed economy it is shown that maximum sustainable debt robustly occurs where the consumption of individual households reaches zero, the limit of its feasible range. Taxation absorbs all of the household's labour income here. In a closed economy, at this point the real interest rate also hits a "ceiling" given by a simple combination of preference parameters and the death probability.

Keywords

maximum sustainable government debt, overlapping generations, uncertain lifetimes

JEL Classification

E62, H63

1. Introduction

In western countries, worries about the maximum level of government debt which a country can sustain have again come to the fore in the wake of the financial crisis of 2008 and of the Eurozone debt crisis to which it gave rise in 2010. In emerging market countries, high government debt and defaults thereon have been a recurring feature for much longer, as for example with the 1998 Russian crisis or the 2001 Argentine crisis. Although there are many studies of the effects of government debt and deficits on the economy, there are few which have directly addressed the question “What is the maximum level of government debt which a country can sustain?” In practice, this limit must be mainly a political one: some interest groups will be more severely affected by the consequences of high government debt than others, and, depending on their political power, such groups will sooner or later succeed in changing policy in order to halt, or reverse, the increase in debt. They may even force the government to default. Nevertheless, what drives the political pressures are the economic effects which debt has. It is therefore desirable to study what are the underlying economic limits to the level of government debt. Specifically, we ask whether there is a level of government debt above which no steady-state equilibrium of the economy exists; and, if so, what determines it. Such a theoretically-derived limit is inevitably going to be higher – perhaps much higher – than the practical limit which applies once political forces are taken account. However, since it provides a “backstop” to any practical limit, it is surely useful to know what it is.

In this paper we investigate this question using a well known and very tractable model which allows government debt to have real effects on the economy: namely, the “perpetual youth” model of Blanchard (1985). In Blanchard’s model, which builds on the earlier work of Yaari (1965), agents have a constant probability of death, and thus a finite expected lifetime.¹ This is a way of incorporating overlapping generations (OLGs) which allows us to parameterise the average length of life by varying the probability of death, unlike in the well-

¹ It is referred to as the “perpetual youth” model by Blanchard and Fischer (1989) because the assumption of a constant probability of death implies that the expected remaining lifetime of an individual is always the same, and so independent of the individual’s age.

known OLG model by Diamond (1965) in which agents live for exactly two periods. Blanchard himself used the perpetual youth model to study the effects of changes in government debt. However he did not consider the model's implications for the *maximum* level of debt which the economy could support, as we do here.

Below we combine a household sector based on a perpetual youth structure with a variety of alternative assumptions about the rest of the economy. As in Diamond (1965) and Blanchard (1985), our focus is on the long run, so that capital accumulation is in general allowed for, prices are flexible and markets are competitive. The economy may be either open or closed. If it is open, then the real interest rate is fixed by the exogenous world interest rate. If it is closed, then the real interest rate is endogenous. The production technology, in which labour and capital are inputs, is allowed to take a fairly general form, and as a special case we also consider pure-exchange economies where there is no production, output being treated as an exogenous endowment.

Our main finding is that in the perpetual youth framework maximum sustainable government debt always occurs at a “degeneracy”, defined as where at least one variable (other than debt) reaches the limit of its feasible range. In particular, at such a point the economy hits the limit of its “taxable capacity”. Here the taxation necessary to finance the debt interest cannot be increased further without causing agents to be insolvent. Individuals' consumption is driven to zero. If government debt were any higher, individuals' consumption would have to be negative for a steady state equilibrium to exist; but negative consumption is infeasible. The result that maximum sustainable debt occurs at a “degeneracy” is very robust in the perpetual youth model: it holds irrespective of whether the economy is open or closed; irrespective of the exact form of preferences and production technology; and irrespective of whether there is production at all. The “degeneracy” result is also – at least, for a closed economy – notably different from the result which holds in Diamond's OLG model. In that model, as was shown in Rankin and Roffia (2003), maximum sustainable debt occurs at an “interior maximum”, where variables are not at the limit of their feasible ranges. In the case of an “interior maximum”, to increase debt one step further is like walking off a cliff: the economy is forced onto an unstable dynamic path, leading over time to complete collapse. In

the case of a “degeneracy”, by contrast, the collapse has already occurred at the point at which the maximum is reached. Our second finding is that the value of the maximum sustainable debt: GDP ratio is determined in a very simple way in this model. We show that it equals the wage share in output divided by the relevant interest rate. Although it is not our aim here to make serious quantitative estimates, crude calculations using this formula suggest that the likely value for the maximum sustainable debt: GDP ratio would be of the order of 10 or more, which is far above any current real-world debt: GDP ratios. Our third finding is that, in a closed economy (in which, unlike in a small open economy, the real interest rate is not exogenous), an equivalent way of describing how maximum sustainable debt is reached is to say that it occurs where the interest rate hits a “ceiling”. The value of the ceiling is given by a simple formula involving the death probability and preference parameters. We show why the ceiling arises by analysing the cross-section composition of household consumption in the steady state.

A number of other studies in the literature have examined the “sustainability” of fiscal policy. However, different authors define this term in different ways. Amongst those who define sustainability in terms of the existence of a steady state, the majority have been more interested in the sustainability of deficits rather than debt. Two interesting recent contributions in this spirit, in the context of endogenous growth, are by Brauning (2005) and Yakita (2008). Yet, even when there is no deficit, a fiscal situation can be unsustainable simply because the inherited stock of debt is too high. This is the problem on which we focus here. Within the Diamond OLG framework, this type of unsustainability was examined for a closed economy by Rankin and Roffia, *op.cit.*, and for a two-country world by Farmer and Zotti (2010).

The structure of the the paper is as follows. In Section 2 we present the building blocks of the economy, centred around the perpetual youth framework. In Section 3 we then study maximum sustainable debt in a small open economy version of the model. Section 4 performs a similar analysis for a closed economy. The broader significance of our results is discussed in Section 5.

2. The Structure of the Economy

As in Blanchard (1985) we assume time is continuous and that an agent has a constant probability of death, p , per unit of time, where $p \geq 0$. The population is constant and its size is normalised to unity. Thus at any point in time, p agents die and p are born. An agent has zero financial wealth at birth and may accumulate or decumulate wealth throughout life. There will hence generally be a distribution of wealth and consumption levels by age across the population. We will index agents by their birthdate, s , so that $c(s,t)$ denotes the consumption at time t of an agent born at time s (and similarly for other variables).

The lifetime expected utility maximisation problem of an agent, from the perspective of time t , can be written as:

$$\text{maximise} \quad \int_t^{\infty} (1-\gamma)^{-1} [c(s,v)]^{1-\gamma} e^{-(\theta+p)(v-t)} dv \quad (1)$$

$$\text{s.t.} \quad da(s,v)/dv = [r(v)+p]a(s,v) + y^N(v) - c(s,v) \quad \text{for all } v \geq t.$$

Flow utility takes the “constant elasticity of intertemporal substitution” form, where $1/\gamma (\geq 0)$ is the elasticity of intertemporal substitution. It is discounted at the sum of the agent’s pure time preference rate, $\theta (\geq 0)$ and death probability, p (noting that $e^{-p(v-t)}$ is the probability density of still being alive at date v). $a(s,v)$ is the agent’s financial wealth at date v , which earns real rate of interest $r(v)$. As in Blanchard (1985), perfectly competitive insurance companies provide an annuity scheme whereby an agent who remains alive from one instant to the next earns, in addition to interest, an annuity at the rate p per unit of wealth; while if the agent dies all their wealth is ceded to the insurance company. This yields actuarially fair insurance and in equilibrium the insurance companies make zero profits. $y^N(v)$ is the net-of-tax non-interest income of the agent. In an economy with production this equals the real wage less a lump-sum tax, $\omega(v) - \tau(v)$, , since agents have one unit of time which they supply completely inelastically to the labour market. In an economy with no production it equals an exogenous income endowment less the lump-sum tax, $\bar{y} - \tau(v)$.

Solving the above optimisation problem yields the Euler equation for optimal intertemporal consumption choice by the individual:

$$[1/c(s,v)]dc(s,v)/dv = (1/\gamma)[r(v) - \theta]. \quad (2)$$

An individual's optimal consumption at any date t can also be expressed as:

$$c(s,t) = w(s,t) / \Delta(t), \quad (3)$$

where

$$w(s,t) \equiv a(s,t) + h(t),$$

$$h(t) \equiv \int_t^\infty y^N(v) e^{-\int_t^v [r(z)+p]dz} dv,$$

$$\Delta(t) \equiv \int_t^\infty e^{\int_t^v [(1/\gamma-1)r(z)-\theta/\gamma-p]dz} dv.$$

Here, $w(s,t)$ is total lifetime wealth, which is the sum of financial wealth $a(s,t)$ (“assets”) and of “human wealth”, $h(t)$. The latter is defined as the present value of current and expected future net-of-tax non-interest income. Since we assume non-interest income and taxation do not depend on age, $h(t)$ is also independent of age, i.e. of the birthdate, s . $\Delta(t)$ is the inverse of the propensity to consume out of total wealth. It depends on current and expected future real interest rates, except when $\gamma = 1$ (which corresponds to the case of a logarithmic utility function), when it reduces to $1/(\theta+p)$.

We next turn from individual to aggregate consumption. We will generally denote aggregate variables in the upper case and the corresponding individual variables in the lower case. Aggregating across all individuals alive at time t by birthdate, s , gives:

$$C(t) \equiv \int_{-\infty}^t p e^{-p(t-s)} c(s,t) ds. \quad (4)$$

Notice that the individual consumption function, (3), is linear in wealth and its form does not depend on the agent's birthdate. Hence an identical relationship applies between the corresponding aggregate variables:

$$C(t) = W(t) / \Delta(t). \quad (5)$$

In the case of the Euler equation we cannot simply replace individual by aggregate variables. Nevertheless we can still derive an “aggregate” counterpart of (2), which is as follows:

$$dC(t)/dt = (1/\gamma)[r(t) - \theta]C(t) - pA(t)/\Delta(t). \quad (6)$$

Compared with (2), the difference is that, for $p \neq 0$, there is an additional, negative, effect in the equation arising from aggregate financial wealth, $A(t)$. A similar equation is derived by Blanchard (1985).

When there is production, we assume that labour and capital are combined to produce output through a production function with constant returns to scale. Since aggregate labour input has been normalised to unity, we can represent technology simply by:

$$Y = F(K), \quad (F' > 0, F'' < 0)$$

where the labour input variable has been suppressed. Profit maximisation and competitive markets yield the usual factor-price conditions:

$$r = F'(K), \quad \omega = F(K) - KF'(K). \quad (7)$$

Lastly, consider the government's budget constraint. We shall abstract from government spending in the sense of purchases of output. Taxation, as noted, takes the form of an equal lump-sum tax, τ , on each agent. Since the population is of size one, aggregate and individual taxation are equal: $T = \tau$. The government will normally balance its budget, so that, letting D denote the outstanding stock of government debt, we have:

$$T = rD. \quad (8)$$

A balanced budget is appropriate since we shall focus on steady states of the model and, without a source of permanent growth, the stock of government debt must be held constant over time in order to have a steady state. However, we shall permit occasional one-off increases or decreases in the stock of debt, implemented through an equal tax cut or tax increase on each agent, with a corresponding discontinuous jump in D . At such points the government will run a deficit or surplus for an instant and (8) will momentarily be violated.

3. A Small Open Economy

A country which can trade freely in world goods and capital markets and is of negligible size relative to the rest of the world (hence “small”), faces an exogenous world real interest rate, r^* . Together with the factor-price conditions (7), this ties down the country’s capital stock, real wage and output levels. We can denote these as:

$$\bar{K} \equiv F'^{-1}(r^*), \quad \bar{\omega} \equiv F(\bar{K}) - \bar{K}F'(\bar{K}), \quad \bar{Y} \equiv F(\bar{K}), \quad (9)$$

where $F'^{-1}(r)$ ($=K$) denotes the inverse of the function $F'(K)$. The given r^* thus effectively makes all the variables on the production side of the economy exogenous. Clearly, then, these are not affected by government debt.

In the open economy, the aggregate financial wealth of domestic households, A , has three components:

$$A = \bar{K} + J + D. \quad (10)$$

These are, namely, the domestic capital stock (\bar{K}), domestic government debt (D), and net foreign assets held by households (J). The latter are claims on foreigners which are accumulated or decumulated as a result of surpluses or deficits in the current account of the balance of payments. The balance-of-payments equation states this:

$$\dot{J} = r^*J + F(\bar{K}) - C. \quad (11)$$

Here, $F(\bar{K}) - C$ is the trade surplus (excess of domestic production over absorption), r^*J is net interest receipts from abroad, and $\dot{J} \equiv dJ/dt$. The right-hand side is hence the current account surplus. Note that the constancy of K implies that investment is zero.²

To study the behaviour of aggregate consumption, we shall use the “aggregate Euler equation”, (6). Since $r(t) = r^*$ for all t , $\Delta(t)$, the inverse propensity to consume out of wealth, becomes exogenous: the relevant formula for it is given below. In the case of the small open economy, (6) can thus be expressed as:

² This is true except in situations in which r^* or $F(\cdot)$ change, which we shall not be concerned with here. We also abstract from depreciation of capital.

$$\dot{C} = (1/\gamma)[r^* - \theta]C - (p/\Delta)[\bar{K} + D + J], \quad (12)$$

$$\text{where } 1/\Delta = p + (1/\gamma)\theta + (1 - 1/\gamma)r^*.$$

The small open economy macroeconomic model hence consists of the pair of differential equations (11) and (12). These jointly govern the evolution of J and C . This is the same as in Blanchard (1985). Note that although K and Y are fixed by r^* , the country's aggregate consumption and also its GNP ($\equiv \bar{Y} + r^*J$) are able to vary, because the country's residents are able to run up or run down their stock of net foreign assets.

Our interest here is in steady states. In a steady state of this economy, the current account must be in balance, so that $\dot{J} = 0$. Imposing this on (11) reminds us that $C = \bar{Y} + r^*J$ in a steady state, showing that steady state consumption is higher or lower as net foreign assets are respectively higher or lower. This is because higher net foreign assets mean higher interest receipts, which enable home residents permanently to consume more.

Government debt has a negative effect on steady state consumption in this economy. The reason is that, although in a small open economy such debt does not crowd out capital, it does crowd out net foreign assets. We can see this by setting $\dot{J} = \dot{C} = 0$ in (11) and (12) and solving for C :

$$C = \frac{p[\theta + \gamma p + (\gamma - 1)r^*]}{(p + r^*)(\theta + \gamma p - r^*)} \{F(\bar{K}) - r^*[\bar{K} + D]\}. \quad (13)$$

(13) shows that the long-run effect of an increase in D on C is negative, since the expression multiplying $\{.\}$ is positive.³ Intuitively, the mechanism at work is as follows. Government debt provides domestic households with an alternative to net foreign assets as a form in which to hold their financial wealth. With more government debt available, they need to hold a smaller amount of net foreign assets in order to have sufficient financial wealth for shifting consumption from earlier to later in their lives.⁴ Since the country has less net foreign assets,

³ Here we assume that the exogenous parameters (θ, γ, p, r^*) are such that $\theta + \gamma p - r^* > 0$. In fact this condition must be satisfied for a valid steady state equilibrium of the model to exist at all: see Section 4.

⁴ Indeed, the steady state equilibrium level of J could be negative, i.e. the country could be a net debtor. In this case an increase in government debt just increases domestic residents' foreign debt.

however, it also receives less interest from abroad, and therefore aggregate steady state consumption must be lower.

Let us now consider what happens to steady state values as the stock of government debt becomes very large. (13) shows that C is a decreasing linear function of D . It is therefore obvious that there exists a critical value of D which drives C to zero. A negative value of C makes no economic sense, so the level of D at which C becomes zero is the “maximum sustainable” level of government debt (as we defined it in the Introduction) for this economy. If D is greater than this critical value, no steady state equilibrium of the economy exists. What we see, then, is that the point at which a steady state equilibrium ceases to exist is a “degeneracy”, by which we mean that, at such a point, at least one economic variable (C , in this case) has reached the limit of its feasible range, so that the steady state equilibrium is “degenerate”. The economy has, in effect, collapsed in the steady state corresponding to this value of D . An alternative way in which the maximum sustainable level of debt could in principle be reached, if the economy were different, is at an “interior maximum”. This has the property that no variable has reached the limit of its feasible range. We will discuss this second type of limit further when we look at the closed economy. For a small open economy, the conclusion that the limit to debt is reached at a “degeneracy” also holds when the Diamond OLG structure is used, as Rankin and Roffia (2003) note.

To understand better what happens when D reaches its maximum sustainable value in this economy, let us set C to zero in (13). Clearly, the value of D at which this occurs must satisfy:

$$\begin{aligned} r^* D &= F(\bar{K}) - r^* \bar{K} \\ &= \bar{\omega} \end{aligned} \tag{14}$$

(where the second line follows from (7)). This shows that, at the point where the limit to debt is reached, interest payments on government debt – and thus, from the government’s budget constraint, (8), the tax, τ , faced by a household – must equal the real wage. Suppose government debt is increased in discontinuous small steps, and between the increases the

economy is allowed to converge to the corresponding new steady state, where one exists.⁵ As this happens, the tax burden on households steadily increases, since higher taxes are necessary to finance the government's higher interest payments. Eventually there comes a point at which all of a household's labour income is taken in taxation. In terms of the notation in (1), the household's net-of-tax non-interest income, $y^N = \bar{\omega} - \tau$, hits zero. In such a steady state, a household has zero "human wealth", since human wealth (as defined in (3)) in a steady state is:

$$y^N / (r^* + p), \quad (15)$$

which is obviously then also zero. This means that a newborn household, which by assumption has no financial wealth, has zero total lifetime wealth. It can therefore afford no consumption. Any further small increase in government debt would cause τ to exceed $\bar{\omega}$, and thus would make newborn households insolvent. Hence another way of describing how the maximum sustainable level of debt is reached, in this "degeneracy" case, is to say that the economy has arrived at the limit of its "taxable capacity". The government is extracting as much tax revenue as it possibly can from households.

This perspective also helps to explain a simple formula which we can derive for maximum sustainable debt. From (14), the limit to government debt can be expressed as:

$$D = \bar{\omega} / r^*. \quad (16)$$

Recalling that the population, and thus aggregate labour supply, is of size one, this says that maximum sustainable debt equals total labour income divided by the world interest rate. (Equivalently, the maximum sustainable debt: GDP ratio equals the wage share in GDP divided by the world interest rate.) Intuitively, this is because, using the government budget constraint, (8), government debt in a steady state can be written as $D = T/r^*$, reminding us that debt equals the present value of tax revenue; while in turn the maximum feasible tax revenue, as just seen, equals total labour income.

We can summarise these findings about the small open economy by:

⁵ Blanchard (1985) discusses the dynamic convergence process. For brevity, we omit analysis of this here.

Proposition 1 *In a small open economy version of the perpetual youth model, maximum sustainable government debt occurs where individual and aggregate consumption are driven to zero and wage income is completely absorbed by taxation. The maximum sustainable debt:GDP ratio is given by the wage share in GDP divided by the world interest rate.*

4. A Closed Economy

In a closed economy, the real interest rate becomes endogenous. Consequently the capital stock and output are also endogenous. On the transition path between steady states, investment will now be non-zero.

The equations which describe the macroeconomic equilibrium are as follows. First, the economy's income-expenditure identity tells us that investment equals output minus consumption:

$$\dot{K} = F(K) - C. \quad (17)$$

Aggregate financial assets consist just of capital and government debt, since there are no net foreign assets: $A = K + D$. Hence the "aggregate Euler equation", (6), can be written:

$$\dot{C} = (1/\gamma)[F'(K) - \theta]C - p(K + D)/\Delta, \quad (18)$$

(in which we have also used (7)). Δ , the inverse propensity to consume out of wealth, is time-varying in the closed economy since the interest rate is time-varying. By differentiating the definition of Δ with respect to t we can derive an equation for its rate of change:

$$\dot{\Delta} = [p + \theta/\gamma + (1 - 1/\gamma)F'(K)]\Delta - 1. \quad (19)$$

The closed-economy macroeconomic model thus consists of the three simultaneous differential equations (17)-(19), in (K, C, Δ) . This is the same as in Blanchard (1985).

As before, we will focus on steady states. Setting $\dot{K} = \dot{C} = \dot{\Delta} = 0$ in (17)-(19), and rearranging the resulting equations to eliminate the steady state values (C, Δ) , we can reduce the system to an equation in the steady state value of K alone:

$$(1/\gamma)[F'(K)-\theta]F(K) = p[p+\theta/\gamma+(1-1/\gamma)F'(K)][K+D]. \quad (20)$$

This determines K as an implicit function of D . Since K has a unique negative relationship with r via $r = F'(K)$, we can equivalently re-express (20) as an equation which implicitly determines r :

$$(1/\gamma)[r-\theta]F(F'^{-1}(r)) = p[p+\theta/\gamma+(1-1/\gamma)r][F'^{-1}(r)+D]. \quad (21)$$

Let us now study the relationship (21) in more detail. We would expect it to imply that an increase in D causes an increase in r , and thus a decrease in K . This is the standard result - found also in Blanchard (1985), and indeed in Diamond (1965) - that, in a closed economy, higher government debt raises the steady state real interest rate and crowds out the private capital stock. We will confirm this below. However, here we are not just interested in a small change in debt; we also wish to study potentially very large changes. To do this we need to understand not only the “local” properties of the relationship (21), but also its “global” properties, for a wide range of values of D .

We cannot obtain a closed-form solution for r as a function of D from (21), but we can obtain a closed-form solution for D as a function of r :

$$D = \frac{(1/\gamma)[r-\theta]}{p[p+\theta/\gamma+(1-1/\gamma)r]} F(F'^{-1}(r)) - F'^{-1}(r). \quad (22)$$

Notice that the first right-hand term corresponds to A and the second to K . Hence (22) says that, for any given steady state value of r , the level of government debt which supports that value equals the difference between households’ demand for financial assets and firms’ demand for physical capital at that value of r . Equivalently, since households’ “demand for financial assets” could also be termed their “supply of financial capital”, (22) says that the government debt which supports a given steady state r equals the private sector’s “excess supply” of capital at that value of r .

Before analysing the function (22) further, we digress to highlight a crucial point. We summarise this as:

Lemma 1 *In a closed economy version of the perpetual youth model, a steady state equilibrium cannot exist unless the interest rate is less than the critical value $\theta + \gamma p$ (the “ceiling” value), where θ is the pure time preference rate, p is the probability of death per unit time, and γ is the inverse of the elasticity of intertemporal substitution of consumption.*

To see why this holds, consider the problem of aggregating individual households’ consumption levels in a steady state. The general relationship of aggregate to individual consumption is given by (4). Now, in a steady state, integrating the Euler equation for an individual household, (2), we readily obtain:

$$c(s,t) = c(s,s)e^{(1/\gamma)(r-\theta)(t-s)}. \quad (23)$$

This says that, for a household born at date s , consumption at date t will have grown relative to consumption at birth by a factor which depends on the age of the household, $t-s$, and on the growth rate of individual consumption, $(1/\gamma)(r-\theta)$. Note moreover that, in a steady state, the cross-section distribution of consumption by age must be constant over time. This means that consumption of the newborn should not depend on the date at which they are born, i.e. $c(s,s)$ should be independent of s . Let us call this value $\overline{c(s,s)}$. Then by substituting (23) into (4) we can calculate aggregate consumption as:

$$C(t) = \overline{c(s,s)}p \int_{-\infty}^t e^{[(1/\gamma)(r-\theta)-p](t-s)} ds, \quad (24)$$

where to have factored out $\overline{c(s,s)}$ is valid because of its independence from s . It is now clear that $C(t)$ can only take a finite value if the value of the integral in (24) is finite. In turn, the integral in (24) will only be finite if $(1/\gamma)(r-\theta) - p$ is negative. Hence the condition for aggregate consumption in a steady state to be finite is that:

$$r < \theta + \gamma p. \quad (25)$$

This is the source of the ceiling on the value of the interest rate.

The intuitive explanation for the interest rate ceiling is as follows. If the interest rate is large, consumption across the population increases rapidly with age. In calculating aggregate

consumption by summing across the population in ascending order of age, the increase which occurs with age is in general offset by the decline in the size of any given age cohort with age, and this is enough to keep aggregate consumption finite. However, if r is too large, the death rate, p , is insufficient to stop the total from “exploding” as we sum across older and older age cohorts.⁶

Let us now return to the function (22). A special case of (22) which it is helpful to consider first is where there is no production. Thus, imagine that output is just an exogenous endowment, \bar{Y} . In this pure exchange economy, government debt can still affect the distribution of consumption between agents of different ages and thus the real interest rate; but it cannot affect aggregate consumption. In (22), output is in general given by $F(F'^{-1}(r))$, so in this special case this term is replaced by \bar{Y} . Moreover, in the absence of physical capital, the second right-hand term vanishes. (22) then reduces to:

$$D = \frac{(1/\gamma)[r-\theta]}{p[p+\theta/\gamma+(1-1/\gamma)r]} \bar{Y} \\ \equiv \phi(r)\bar{Y}. \quad (26)$$

What are the properties of the function $\phi(r)$? When $\gamma = 1$ (i.e. utility of consumption is logarithmic), $\phi(r)$ is clearly linear and increasing in r . For $\gamma \neq 1$, on the other hand, the graph of $\phi(r)$ is a rectangular hyperbola. Its exact shape depends on whether $\gamma > 1$ or $\gamma < 1$. We sketch it in Figure 1: panel (a) for $\gamma > 1$ and panel (b) for $\gamma < 1$. By closer inspection of (26), it is straightforward to see that in all cases $\phi(r)$ must be upward-sloping and cut the horizontal axis at $r = \theta$. When $\gamma > 1$, it is continuous for all $r \geq \theta$ and tends to a horizontal asymptote at $1/p(\gamma-1)$. When $\gamma < 1$, over the range $r \geq \theta$ it is discontinuous at, and tends to a vertical asymptote at, $r = (\theta + \gamma p) / (1 - \gamma)$.

⁶ Blanchard (1985) also acknowledges this ceiling on the interest rate at the end of section II of his paper. However he merely records it in passing. It does not play a significant role in his main results.

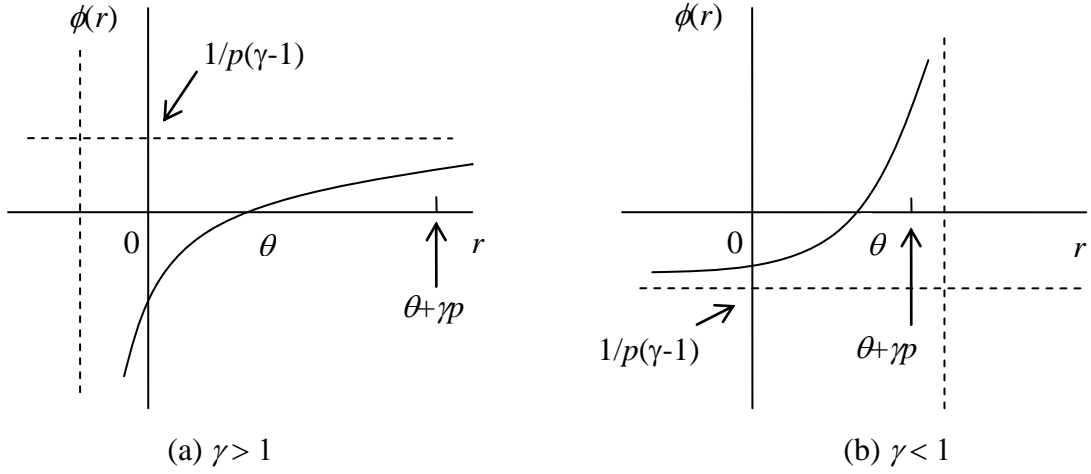


Figure 1 Graphs of the function $\phi(r)$

To consider the maximum sustainable government debt in this pure exchange economy, we now ask: what is the value of r that maximises D , subject to the constraint that a steady state equilibrium exists? The latter constraint means that the interest-rate ceiling (25) must be respected. The answer is immediate from Figure 1: regardless of whether $\gamma > 1$, $\gamma = 1$ or $\gamma < 1$, $\phi(r)$, and thus D , is maximised at the “ceiling” value of r , i.e. where $r = \theta + \gamma p$. (In the case $\gamma < 1$, notice that $\theta + \gamma p$ lies to the left of the vertical asymptote.)⁷

There is another way of seeing what happens as the economy approaches its maximum sustainable debt level. So long as $r < \theta + \gamma p$ holds, (24) can be evaluated as:

$$C(t) = \frac{\gamma p}{\theta + \gamma p - r} \overline{c(s, s)}. \quad (27)$$

We would of course expect the coefficient of proportionality between “newborn” consumption and aggregate consumption to be positive, and this is clearly true when $r < \theta + \gamma p$. We can also see that for $r > \theta + \gamma p$, it becomes negative. A negative value does not make sense, since neither $C(t)$ nor $\overline{c(s, s)}$ can be negative. The explanation is that, when $r > \theta + \gamma p$, (27) is incorrect as the evaluation of the integral in (24), because the latter is then infinite. Now consider what happens as r approaches $\theta + \gamma p$ from below. The coefficient in

⁷ Since the constraint (25) defines the permissible values of r as an “open” interval, we should strictly speaking refer to the “supremum”, rather than the “maximum”, of D . However the use of more ordinary language does not cause confusion here.

(27) tends to infinity. However, in general equilibrium, $C(t)$ equals the exogenous \bar{Y} . Hence, in the general equilibrium solution, newborn consumption is driven to zero as r approaches $\theta + \gamma p$. Intuitively, this must occur because, in a cross-section of the population, consumption rises very quickly with age when r is large. To keep aggregate consumption equal to the given output, consumption of the newborn must shrink, since at given r this shrinks consumption at all ages (see again (23)). As r approaches the critical “ceiling” value, this argument requires newborn consumption actually to shrink to zero.

If newborn consumption is zero at the interest-rate ceiling then human wealth must also equal zero there, which in turn means that taxation must completely exhaust the household’s non-interest income. This is the same as in the small open economy. The reasoning, as before, is that the newborn have no financial wealth and so their consumption is given by:

$$c(s, s) = h(s) / \Delta . \quad (28)$$

(This has been obtained from the individual’s consumption function, (3), with Δ here taking its steady state value of $[p + (1/\gamma)\theta + (1 - 1/\gamma)r]^{-1}$). Hence $c(s, s) = 0$ implies $h(s) = 0$. $h(s)$ in the steady state of the pure exchange economy is $(\bar{y} - \tau) / (r + p)$ (cf. (15)), where \bar{y} ($= \bar{Y}$) is the individual’s endowment income. Thus, as D is increased and r approaches its ceiling value, τ ($= T = rD$) must approach \bar{y} in order to drive $h(s)$ to zero. Therefore we see that in this version of the closed economy, just as in the small open economy, at the maximum sustainable government debt the economy reaches the limit of its taxable capacity. As in the small open economy, the maximum occurs at a “degenerate” steady state, where newborn consumption is driven to its lowest feasible value, namely zero. (A difference from the small open economy, on the other hand, is that *aggregate* consumption does not tend to zero.)

Having analysed the special case of a pure exchange economy, we return to the economy with production. Using (7) and the definition of $\phi(r)$ in (26), we can rearrange the function (22) as:

$$D = \phi(r) \{ \omega(r) - [1/\phi(r) - r] F'^{-1}(r) \}, \quad (29)$$

where $\omega(r) [\equiv F(F'^{-1}(r)) - rF'^{-1}(r)]$ gives the real wage as a function of r . Compared to its counterpart (26) in the pure exchange economy, (29) replaces the exogenous \bar{Y} term by $\{.\}$, which is another function of r . We already know that $\phi(r)$ is unambiguously increasing in r for all r in the interval $[\theta, \theta + \gamma p]$. Consider now how the term $\{.\}$ varies with r over this interval. $F'^{-1}(r)$ (firms' demand for capital) is decreasing in r , while $\omega(r)$ is easily shown to be increasing.⁸ It is also clear that the term $[1/\phi(r) - r]$ is decreasing in r . Given that $[1/\phi(r) - r]$ is moreover always positive⁹, we can then see that $\{.\}$ is unambiguously increasing in r . (29) therefore consists of the product of two positively-valued functions, both of which are increasing in r , for r in the range of interest. It follows that, in the economy with production, it is still the case that D is unambiguously increasing in r , for r in the relevant range. Therefore, just as in the pure exchange economy, government debt in the economy with production reaches its maximum sustainable value at the interest rate ceiling.¹⁰

In the economy with production, output declines as government debt increases because the economy's capital stock declines. This is obvious from the fact that $K = F'^{-1}(r)$. Note, however, that K and Y do not in general tend to zero as the maximum sustainable debt is approached. Rather, they tend to $K = F'^{-1}(\theta + \gamma p)$ and $Y = F(F'^{-1}(\theta + \gamma p))$, which will generally be positive. Nevertheless, as in the pure exchange economy, as the interest rate ceiling is approached, the consumption of newborn agents is driven to zero and all of a household's non-interest income (which, here, means wage income) is taken in taxation. The arguments that we used above to show this for the pure exchange economy clearly still apply here. Maximum sustainable debt therefore continues to occur at a "degeneracy".

These results for the closed economy contrast with those obtained using Diamond's (1965) OLG framework. As Rankin and Roffia (2003) showed, in Diamond's framework maximum sustainable debt is reached at an "interior maximum", not a "degeneracy". In the case of an interior maximum, the counterpart of the function (22) is non-monotonic in r over the feasible range of r , so that the value of r at which D reaches a maximum is not at the limit

⁸ At this point let us further assume that $F(\cdot)$ satisfies the "Inada conditions" that $\lim_{K \rightarrow 0} F'(K) = \infty$, $\lim_{K \rightarrow \infty} F'(K) = 0$. This will ensure that $F'(K) = r$ has a solution for all $r \in [\theta, \theta + \gamma p]$.

⁹ $1/\phi(r) - r = (r + p)(\theta + \gamma p - r) / (r - \theta)$, which is positive for all $r \in (\theta, \theta + \gamma p)$.

¹⁰ Again, the qualification in footnote 7 applies.

of the feasible range of r . Nor is it at the limit of any other variable associated with the steady state, which also means that the economy is not at the limit of its taxable capacity. In such a situation, the consequence of a further small increase in D is to force the economy onto an unstable time path. It hence triggers an abrupt change in the local dynamic behaviour of the model (a “catastrophe”). This path will indeed lead to a degenerate outcome, i.e. a collapse of the economy; but this outcome is only reached after a certain interval of time, and with further, possibly large, changes in some variables.

It should be emphasised that, neither in the closed-economy perpetual youth model nor in the closed-economy Diamond model, is the nature of the maximum sensitive to the exact specification of households’ preferences or firms’ technology. In this paper we have used the fairly general “constant elasticity of intertemporal substitution” utility function, (1), and we have seen that the value of this elasticity, $1/\gamma$, is not relevant for the result. We have also seen that the shape of the production function, $F(\cdot)$, is not critical for the result. Indeed, we showed that the nature of the maximum is unaffected by whether there is production at all.

As in Section 3, we can derive a simple expression for the maximum sustainable debt level. Since the latter occurs where wage income is entirely absorbed by taxation, the formula (16) still applies, except that the “world” interest rate used in (16) is now replaced by the “ceiling” interest rate. Denoting $\theta + \gamma p$ as \hat{r} , and the associated steady state values of other variables as \hat{Y} , etc., we have:

$$D/Y = \hat{\omega} / \hat{r} \hat{Y}. \quad (30)$$

This says that the maximum sustainable debt: GDP ratio equals the wage share in GDP evaluated at the maximum sustainable debt level, $\hat{\omega} / \hat{Y}$, divided by the ceiling interest rate, \hat{r} . A very simple case is where the wage share in GDP is independent of the debt level. This arises with a Cobb-Douglas production function such as $Y = K^\alpha$ ($0 < \alpha < 1$). In this case (30) reduces to:

$$D/Y = (1 - \alpha) / (\theta + \gamma p). \quad (31)$$

For the Cobb-Douglas case it is clear that any change in an exogenous parameter which raises the ceiling interest rate reduces the maximum sustainable debt: GDP ratio. Hence the maximum sustainable debt: GDP ratio is reduced by greater “impatience” by households (higher θ), by lower intertemporal substitutability of consumption (higher γ), and by lower life expectancy (higher p).¹¹

Our findings for the closed economy, then, can be summarised as:

Proposition 2 *In a closed-economy version of the perpetual youth model, maximum sustainable government debt occurs where individual consumption is driven to zero and wage income is completely absorbed by taxation. Equivalently, it occurs where the interest rate reaches its “ceiling” value. The maximum sustainable debt: GDP ratio is given by the wage share in GDP divided by the ceiling interest rate.*

5. Discussion

As made clear in the Introduction, our aim in this paper has been to explore the limits to government debt in a well-known and tractable model of overlapping generations. The situations studied here are hypothetical extremes. We have acknowledged that, in practice, political constraints on how much debt can be issued will become binding well before the constraints of pure economic feasibility. To avoid cluttering the analysis, we have also abstracted from many “realistic” features of actual economies. In further work, it would be interesting to include more of the latter. For example, a natural extension would be to allow for sources of underlying economic growth, such as population increase or technological progress. Another extension would be to use a utility function which incorporates a “subsistence” level of consumption, which would imply a lowest feasible value of consumption which is strictly positive, rather than zero.¹²

¹¹ Note that we should resist any temptation to apply the formula when $p = 0$. In that case any level of government debt is sustainable, since it has no real effects.

¹² I am grateful to a referee for this suggestion.

For the reasons just given, the model as it currently stands does not permit a serious quantitative assessment of maximum sustainable debt. Nevertheless, the kinds of numbers implied may be worth a quick glance. For the purposes of comparison, it may be helpful to bear in mind actual government debt: GDP ratios which have been experienced. The recent paper and database by Abbas et al. (2010) is helpful here. In their database, the highest debt: GDP ratio ever recorded was 2093% (for Nicaragua, in 1990). In the present day, the distinction of having the highest debt: GDP ratio goes to Japan, with 220% in 2010.¹³ In our model, let us now consider likely values for the parameters in the formula (31). A widely accepted figure for the wage share in GDP, $1-\alpha$, is $2/3$. The pure time preference rate, θ , might be plausibly set at 0.02 (measuring time in years). For the elasticity of intertemporal substitution in consumption ($1/\gamma$), a commonly used number is 1. Less consensus exists over what is a reasonable value for the death probability, p . The expected lifetime of an individual in the model is $1/p$. This might suggest setting $1/p$ at about 70 years, and thus p at, say, 0.014. If we adopt this set of values, (31) yields a maximum sustainable debt: GDP ratio of 19.7. This is clearly far above any value that is currently observed, even if it is not above the extraordinary Nicaraguan number. On the other hand, a case for other parameter values could be made. Some empirical studies of consumption suggest an elasticity of intertemporal substitution of about 0.5, i.e. of $\gamma = 2$. It could also be argued that $1/p$ should not be equated to the expected remaining lifetime of the newborn, but to that of an average member of the population. This would reduce it to about 35 ($p = 0.028$), say. With these alternative values, (31) yields a D/Y ceiling of 8.82. This is still well above current actual debt: GDP ratios.¹⁴

We have noted that the perpetual youth framework implies that maximum sustainable debt is reached in a different way from the way it is reached in the Diamond OLG framework (at least, in a closed economy): namely, at a “degeneracy” not an “interior maximum”. It is

¹³ The accurate measurement of government debt in practice is of course well known to be subject to numerous difficulties. These figures should therefore be treated with caution. Moreover, the figures cited here are generally not “steady state” figures.

¹⁴ A more radical calculation might take p to be, not the probability of “physical death”, but of “economic death”. See Bayoumi and Sgherri (2006) for such an argument. They use this to justify estimating p econometrically on the basis of consumption data. This yields a central estimate for p of about 0.17, and thus an expected “economic lifetime” of 6. If such a value were accepted, then with $\theta = 0.02$ and $\gamma = 2$, the D/Y ceiling would be as low as 1.9.

not possible to say unambiguously which of these analyses is “correct”. As a broad statement, the “interior maximum” which occurs in the Diamond framework seems likely to result from debt having a stronger effect on the real interest rate in that framework than in the perpetual youth framework. This hypothesis is suggested by the following two facts. First, when the real interest rate is pegged to the world interest rate, as happens in a small open economy, then even in the Diamond framework maximum sustainable debt occurs at a “degeneracy”. Second, in the closed-economy perpetual youth framework, we have seen that there is a sharply defined ceiling on the interest rate, whereas no such ceiling exists in the Diamond framework. If we ask which of these two types of OLG structure is more relevant empirically, a couple of remarks may be worth making. On the one hand, the human “life cycle” is arguably captured better by the Diamond model. On the other hand, it is notable that empirical work which has attempted to identify a relationship between government debt and the real interest rate has found that, while such a relationship exists, it is not particularly strong.¹⁵ Evidence of the latter kind, therefore, would tend to favour the perpetual youth framework.

¹⁵ See, for example, Bernheim (1987) or Engen and Hubbard (2004).

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